CONCERNING THE INVARIANT POINTS OF COMMUTATIVE COLLINEATIONS.

By William Benjamin Fite.

1. In his *Geometrie der Lage** Reye states that a space collineation with just four invariant points is commutative only with the ∞^3 collineations that leave these points invariant.

The homogeneous linear substitution that represents such a collineation can have no two of its multipliers equal. Moreover if two collineations are commutative, the corresponding substitutions need not be commutative—one may transform the other into itself multiplied by a similarity substitution. If now in Reye's theorem we understand by commutative collineations those whose corresponding substitutions are commutative, the theorem is a special case of a much more general theorem. Without this restriction the theorem is not true, as I shall show.

If any substitution S transforms the substitution A into itself multiplied by a similarity substitution not identity, the sums of the multipliers of A and of S must be zero.† Reye's theorem applies then if the sum of the multipliers of A is not zero. The exception appears only in the case of collineations with "eingeschriebene Tetraederlage"‡, since a necessary and sufficient condition for such collineations is that the sum of the multipliers be zero.§

We suppose then that A has just four invariant points and that the sum of its multipliers is zero. It can be transformed to the normal form ¶

$$A \equiv \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 0 & \omega_4 \end{pmatrix}, \ (\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0).$$

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* Second part, 3rd edition, pp. 90, 91. Cf. *Schoenflies, Encyclopædie der Mathematischen Wissenschaften*, Band III, Heft 3, p. 468.

† Fite, *Transactions of the American Mathematical Society*, vol. 7 (1906), p. 66.

‡ R. Sturm, *Die Lehre von den geometrischen Vervandtschaften*, vol. III, p. 288.

§ After having proved this for any number of variables I found that the sufficiency of the condition for $n = 3$ had been given by Professor Morley in his lectures at Johns Hopkins University in 1902–3. The necessity of the condition is obvious.

¶ Cf. Weber, *Algebra*, vol. 2 (2nd edition), p. 175.

If S is a substitution such that $S^{-1}AS = AT$, where

$$T \equiv \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \rho \end{pmatrix},$$

ρ must be a fourth root of unity.

We shall treat the cases $\rho = -1$ and $\rho = i$ separately.*

2. $\rho = -1$. We can assume that A is of the form

$$A \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, (\lambda \neq \pm 1).$$

In order that $S^{-1}AS = AT$, where

$$T \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

S must be of the form

$$S \equiv \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & a_{43} & 0 \end{pmatrix}.$$

The only invariant points of S are $(\sqrt{a_{12}}, \sqrt{a_{21}}, 0, 0)$, $(\sqrt{a_{12}}, -\sqrt{a_{21}}, 0, 0)$, $(0, 0, \sqrt{a_{34}}, \sqrt{a_{43}},)$, and $(0, 0, \sqrt{a_{34}}, -\sqrt{a_{43}},)$, provided that $a_{12}\, a_{21} \neq a_{34}\, a_{43}$. If this condition is not satisfied, S has invariant points in addition to these. If we denote the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$, by A_1, A_2, A_3, and A_4 respectively, it is clear that the first two of these invariant points of S lie on the line $A_1 A_2$ and are separated harmonically by A_1 and A_2; and that the last two lie on the line $A_3 A_4$ and are separated harmonically by A_3 and A_4.

* The case $\rho = -i$ is not essentially different from the case $\rho = i$.

Conversely, any point $(\rho_1, \rho_2, 0, 0)$, except A_1 and A_2, on $A_1 A_2$ and its harmonic conjugate $(\rho_1, -\rho_2, 0, 0)$ with respect to these two points, together with any point $(0, 0, \sigma_1, \sigma_2)$ except A_3 and A_4, on $A_3 A_4$ and its harmonic conjugate with respect to these two points are invariant points of collineations that are commutative with A. Such collineations are of the form

$$S \equiv \begin{pmatrix} 0 & \rho_1^2 & 0 & 0 \\ \rho_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1^2 \\ 0 & 0 & \sigma_2^2 & 0 \end{pmatrix}.$$

If the σ's are multiplied by a common factor μ different from unity, the given invariant points are not affected, but we get a different collineation for S. Hence there is a single infinity of collineations commutative with A that have the given points for invariant points. Moreover the invariant points of A are not invariant under these collineations.

It is clear that any point on $A_1 A_2$ or $A_3 A_4$ is transformed by A into its harmonic conjugate with respect to A_1 and A_2 or A_3 and A_4 respectively. Hence: —

THEOREM. *Let A be any three-dimensional collineation with just four invariant points such that the points on two opposite edges of its invariant tetraedron are transformed by it into their harmonic conjugates with respect to the invariant points lying on these edges. Then there is a single infinity of collineations that are commutative with A and that have for invariant points any two points except the vertices on these two opposite edges and the harmonic conjugates of these points with respect to the corresponding vertices. Moreover these collineations do not leave invariant the invariant points of A.**

3. $\rho = i$. In this case $S^{-1}AS = AT$, where

$$T \equiv \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

* It is clear that the collineations A and S of this section can be so chosen that they will both put real points into real points and will both have real invariant tetraedra.

We can assume that

$$A \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

and then, as may easily be verified,

$$S \equiv \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ a_{21} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{43} & 0 \end{pmatrix}.$$

The only invariant points of S are

$$\left(a_{14},\ \frac{a_{21}a_{14}}{a},\ \frac{a_{32}a_{21}a_{14}}{a^2},\ a\right),\quad \left(a_{14},\ \frac{ia_{21}a_{14}}{a},\ -\frac{a_{32}a_{21}a_{14}}{a^2},\ -ia\right),$$

$$\left(a_{14},\ -\frac{a_{21}a_{14}}{a},\ \frac{a_{32}a_{21}a_{14}}{a^2},\ -a\right),\ \text{and}\ \left(a_{14},\ -\frac{ia_{21}a_{14}}{a},\ -\frac{a_{32}a_{21}a_{14}}{a^2},\ ia\right),$$

where $a = \sqrt[4]{a_{14}\ a_{43}\ a_{32}\ a_{21}}$

The first of these points is transformed into the others by A and its powers. Moreover we can determine S uniquely in such a way that the first of these points is any point not lying in a face of the invariant tetraedron of A, and the others are the points into which this point is transformed by A and its powers.*

This form of A is also a special case of the form considered in §2, and therefore there are collineations commutative with A with invariant points such as are described in that section.

These results can be formulated in the

THEOREM. *If A is a collineation with "eingeschriebene Tetraederlage," of period four, and with just four invariant points, which are taken as the ver-*

* It is clear that A is of period 4.

tices of the tetraedron of reference, and if $(\rho_1, \rho_2, \rho_3, \rho_4)$ *is any point not in a face of the tetraedron of reference, then*

$$S \equiv \begin{pmatrix} 0 & 0 & 0 & \rho_1 \\ \frac{\rho_2\rho_4}{\rho_1} & 0 & 0 & 0 \\ 0 & \frac{\rho_3\rho_4}{\rho_2} & 0 & 0 \\ 0 & 0 & \rho_4 & 0 \end{pmatrix}$$

is commutative with A *and has for its only invariant points the point* $(\rho_1, \rho_2, \rho_3, \rho_4)$ *and the points into which this is transformed by* A *and its powers.** *Moreover* S *is the only collineation that is commutative with* A *and that has these invariant points. The invariant points of* A *are not invariant under* S.

4. The general results of §§2 and 3 are applicable to collineations in n variables †$(n \geqq 2)$. Let d $(1 < d \leqq n)$ be any divisor of n and consider any simplex‡ with n vertices that is contained in a linear space of $n-1$ dimensions. We can select n/d linear bounding spaces of this simplex, each of $d-1$ dimensions, in such a way that no two of them have a common point. Then there are $\infty^{\frac{n}{d}-1}$ collineations A of period d that leave invariant the vertices of this simplex and no other points and that, together with their powers, transform any point in any one of these bounding spaces but not in any bounding space of fewer dimensions into d distinct points. If we select any one of these collineations A and a corresponding set of d such points in each of the n/d bounding spaces just referred to, there are $\infty^{\frac{n}{d}-1}$ collineations S that leave these points invariant and that are commutative with A. Moreover none of these collineations leave the invariant points of A invariant.

This result is a generalization of the theorem given by Stéphanos for§ $n = 2$.

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* These points are obviously not co-planar.

† I am indebted to Professor Virgil Snyder for the suggestion to apply the general results of §§2 and 3 to collineations in n variables.

‡ Cf. Schoute, *Mehrdimensionale Geometrie*, vol. 1, p. 9.

§ *Mathematische Annalen*, vol. 22 (1883), p. 313.

5. The collineations commutative with two given commutative collineations. For one of the given collineations we shall take a collineation A in n variables that is of period n and that has just n invariant points. Such a collineation can be written in the form $A:\ x_i' = \omega^{i-1}x_i$ $(i = 1, 2, \cdots, n)$, where ω is a primitive nth root of unity. For the other given collineation we can take any collineation that is commutative with A. We shall consider first the collineation $S:\ x_i' = \rho^i\, x_{i-1}$ $(i = 1, 2, \cdots, n)$, where the subscripts are to be taken *modulo n*. Now a necessary and sufficient condition that a collineation be commutative with A is that it be of the form $x_i' = a_i x_{a+i}$. Hence S is commutative with A.

We consider now any collineation $S_1:\ x_i' = \sigma_i x_{d+i}$, where $0 \leqq d < n$, that is commutative with A, and determine under what conditions it is also commutative with S. Evidently $S_1 S$ is of the form $x_i' = \sigma_i \rho_{d+i}\, x_{d+i-1}$, while SS_1 is of the form $x_i' = \rho_i \sigma_{i-1} x_{d+i-1}$. Hence a necessary and sufficient condition that S and S_1 be commutative is that

$$\frac{\sigma_1 \rho_{d+1}}{\rho_1 \sigma_n} = \frac{\sigma_i \rho_{d+i}}{\rho_i \sigma_{i-1}}$$

for all values of i from 2 to n inclusive. For $i = 2$, we have *

$$\sigma_n = \frac{\rho_2 \rho_{d+1}}{\sigma_2 \rho_{d+2}}.$$

Moreover

$$\sigma_i = \frac{\sigma_2^{i-1} \rho_3 \rho_4 \cdots \rho_i \rho_{d+2}^{i-2}}{\rho_2^{i-2} \rho_{d+3} \rho_{d+4} \cdots \rho_{d+i}}.$$

If in this formula we put $i = n$ and equate the resulting value of σ_n with the one just found, we get just n distinct values for σ_2. But the value of σ_2 determines uniquely the values of all the other σ's. Hence there are exactly n collineations of the form S_1 that are commutative with S. But the n collineations $A^j S^{-d}$ $(j = 0, 1, \cdots, n-1)$ are all of this form and are all commutative with S (since A and S are commutative). We conclude therefore that *the only collineations that are commutative with both* A *and* S *are those contained in the abelian group of order* n^2 *generated by* A *and* S.

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* For convenience we shall put ρ_1 and σ_1 each equal to unity. This obviously puts no restrictions on the collineations S and S_1.

If S is of the form $x'_i = \rho_i x_{d+i}$, where d is relatively prime to n, the result is the same as in the case just discussed. But if d is not relatively prime to n the situation is more complicated. If c is the greatest common divisor of n and d, and $n = rc$, then in order that a collineation S_1 of the form $x'_i = \sigma_i\, x_{e+i}$ be commutative with S it is necessary and sufficient that

$$\frac{\rho_1\sigma_{d+1}}{\sigma_1\rho_{e+1}} = \frac{\rho_{id+j}\sigma_{(i+1)d+j}}{\sigma_{id+j}\rho_{id+e+j}}$$

for all values of i from 0 to $r-1$ inclusive and all values of j from 1 to c inclusive.† Putting $j = 1$, we get

$$\sigma_{d+1}^r = \frac{\rho_{e+1}^{r-1}\rho_{d+1}\cdots\rho_{id+1}\cdots\rho_{(r-1)d+1}}{\rho_{d+e+1}\cdots\rho_{id+e+1}\cdots\rho_{(r-1)d+e+1}}.$$

For $j = 2$ we get

$$\sigma_{d+1}^r = \frac{\rho_{e+1}^r\rho_2\cdots\rho_{id+2}\cdots\rho_{(r-1)d+2}}{\rho_{e+2}\cdots\rho_{id+e+2}\cdots\rho_{(r-1)d+e+2}}.$$

If e is not a multiple of c, these two values of σ_{d+1}^r, when equated, give a relation connecting the coefficients of S.

Therefore in this case S is, in general, not commutative with any collineation of the form S_1.

But if e is a multiple of $c (e = sc)$, we have $\sigma_{d+1}^r = \rho_{sc+1}^r$. This gives r values for σ_{d+1}, and for each of these values there is a single set of values for σ_{id+1} $(i = 1, 2, \cdots, r-1)$. In a similar way we get

$$\sigma_{d+j}^r = \frac{\sigma_j^r\rho_{sc+j}^r}{\rho_j^r},$$

and each of the resulting r values of σ_{d+j} (in terms of σ_j and the coefficients of S) determines a single set of values of σ_{id+j}. Moreover, inasmuch as

$$\frac{\rho_1\sigma_{d+1}}{\sigma_1\rho_{e+1}} = \frac{\rho_j\sigma_{d+j}}{\sigma_j\rho_{e+j}},$$

the value chosen for σ_{d+j} (in terms of σ_j) is uniquely fixed by the value chosen for σ_{d+1}. Hence *there are* ∞^{c-1} *collineations of the form* S_1 *that are commutative with both* A *and* S.

† If $d = r_1c$ and x is a root of the congruence $r_1x \equiv 1 \pmod{r}$, then $id + j \equiv i_1d + c + j$. $\pmod{n}$, where $i_1 \equiv i - x \pmod{r}$. Hence it is unnecessary to give to j values greater than c.

If c_1 is the greatest common divisor of e and n and $n = gc_1$, the order of S_1, if it is finite, is obviously a multiple of g. Now S_1^g is of the form $x'_{ie+j} = \beta_j\, x_{ie+j}$ $(i = 1, 2, \cdots, g;\ j = 1, 2, \cdots, c_1)$, where $\beta_j = \sigma_j\, \sigma_{e+j} \cdots \sigma_{le+j} \cdots \sigma_{(g-1)e+j}$. But $le + j \equiv i_1 d + j_1 \pmod{n}$ for a suitably chosen i_1, when j_1 is the least positive residue of j with respect to the modulus c. Hence $\sigma_{le+j} = \sigma_{j_1} R_l$ and $\beta_j = \sigma_{j_1}^{\,g} T_j$ where T_j and R_l are rational functions of the coefficients of S. For any S_1 whose order divides mg, $\sigma_{j_1}^{mg}$ $(j_1 = 1, 2, \cdots, c)$ is fixed. Moreover there are r choices for σ_{d+j_1} and for each of these choices there are mg values for each σ_{j_1}, except σ_1, which is equal to unity. Therefore *the number of collineations of the form S_1 that are commutative with both A and S and whose orders divide mg is $r(mg)^{c-1}$.*

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